CHARACTERISTIC DIMENSION AND FORM FACTOR OF A SOLID HOMOGENEOUS BODY

N. A. Yaryshev and A. S. Stolyarov

UDC 536.021

Equations for the characteristic dimension and the form factor of a real body are derived on the basis of an approximate geometrical similarity between it and a reference model.

The unified solutions to heat conduction problems [1-3] for bodies of canonical shapes (plate, cylinder, and sphere) can be extended to the case where a body is subject to more than one thermal effect, and can be used for approximately calculating the characteristic temperature parameters (mean-over-the-surface, mean-over-the-volume, and center temperature).

The thermal state of bodies with a three-dimensional temperature distribution was analyzed by a repetitive use of the one-dimensional solution [6, 7]. The bodies under consideration were classified into three groups, their respective principal representatives being an infinitely large plate, an infinitely long cylinder, and a sphere. A real body was compared to each of these models, in order to establish the conditions of approximate geometrical similarity (comparison based on surface area or volume) and physical similarity (comparison based on thermal fluxes or form factors).

An application of these basic principles results in a small error in temperature calculations only when the body is entirely identical to any one of these three canonical models. As a rule, exact and approximate temperatures are compared just for such "favorable" cases with moderate values of the heat transfer coefficient. There are many bodies, however, which do not quite fit into these three basic classes. A typical example is a rectangular parallelepiped with the ratio of sides 1 : 2 : 3.

We propose here a new procedure for establishing the approximate geometrical similarity, which will extend the applicability of one-dimensional solutions to the equation of heat conduction.

We are to determine two geometrical parameters in the unified solutions, namely the characteristic dimension R and the form factor n of a body [4].

A real solid homogeneous body will be characterized by its surface area S, its total volume V, and three orthogonal linear segments (dimensions) $2L_1$, $2L_2$, $2L_3$ (e.g., length, width, and height). Depending on the shape, we will refer the body to one of the three classes represented respectively by a triaxial ellipsoid with semiaxes a, b, c, an infinitely long elliptical cylinder with semiaxes b, c, and an infinitely large (in two directions) plate with the thickness 2c. The problem of classifying a given body will be solved on the basis of its dimensional proportions.

Let us assume that a certain body belongs to the class of ellipsoids. Then its characteristic dimension R and form factor n are determined on the basis of the following considerations:

1. The basic half-dimensions of the body are proportional to the ellipsoid semiaxes

$$\frac{L_1}{c}=\frac{L_2}{b}=\frac{L_3}{a}\,(L_1\leqslant L_2\leqslant L_3;\; c\leqslant b\leqslant a); \tag{1}$$

2. the characteristic dimension of the body is equal to the characteristic dimension of the ellipsoid

$$R=R_E \tag{2}$$

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 24, No. 3, pp. 507-513, March, 1973. Original article submitted April 6, 1971.

TABLE 1. Values of the Function $\Phi(k_{1,2};\ k_{1,3})$

$k_{1,2}$	$k_{1,3}$										
	1	0,9	0,8	0,7	0,6	0,5	0,4	0,3	0,2	0,1	0
1	0,283	0,280	0,275	0,267	0,254	0 237	0,214	0,186	0,152	0,107	0
0,9	0,280	0,280	0,271	0,270	0 258	0,241	0,221	0,190	0,155	0,109	0
0,8	0,275	0,271	0,274	0,269	0,257	0,242	0,219	0,191	0,157	0,110	0
0,7	0,267	0,270	0,269	0,264	0,254	0,239	0,217	0,191	0,155	0,110	0
0,6	0,254	0,258	0,257	0,254	0,246	0,232	0,212	0,185	0 150	0,106	0
0,5	0,237	0,241	0,242	0,239	0,232	0,219	0,199	0,175	0,143	0,100	0
0,4	0,214	0,221	0,219	0,217	0,212	0,199	0,183	0,160	0,130	0,091	0
0,3	0,186	0,190	0,191	0,191	0,185	0,175	0,160	0,140	0,114	0,080	0
0,2	0,152	0,155	0,157	0,155	0,150	0,143	0,130	0,114	0,092	0,064	0
0,1	0,107	0,109	0,110	0,110	0,106	0,100	0,091	0,080	0,064	0,045	0
0	0	0	0	0	0	0	0	0	0	0	0

TABLE 2. Values of the Function $\Phi(k_{1,2})$

$k_{1,2}$	1	0,9	0,8	0,7	0,6	0,5	0,4	0,3	0,2	0,1	0
$\Phi(k_{1,2})$	0,159	0,158	0,154	0,147	0,136	0,121	0,102	0,079	0,054	0,027	0

3. the real body and the ellipsoid have equal surfaces

$$S = S_{\mathrm{E}} \tag{3}$$

4. the geometrical characteristics of the body, as has been shown in [4], are related as follows:

$$\frac{V}{S} = \frac{R}{n+1}. \tag{4}$$

The system of Eqs. (1)-(4) is not closed and, therefore, we solve the additional problem of determining the characteristic dimension R_{E} and the form factor n_{E} of the ellipsoid. Inserting the known expressions for the volume and the surface area of a triaxial ellipsoid into Eq. (4), we find

$$\frac{R_{\mathrm{E}}}{c} = \frac{2}{3} \cdot \frac{n_{\mathrm{E}}+1}{k_{1,2}k_{1,3}f(k_{1,2};\ k_{1,3})}. \tag{5}$$

Here and subsequently $k_{1,2}$ and $k_{1,3}$ denote respectively the following ratios according to (1):

$$k_{1,2} = \frac{c}{b} = \frac{L_1}{L_2},\quad k_{1,3} = \frac{c}{a} = \frac{L_1}{L_3}\ (k_{1,2} \geqslant k_{1,3}), \tag{6}$$

$$f(k_{1,2};\ k_{1,3}) = 1 + \frac{k_{1,3}}{k_{1,2}} \cdot \frac{1}{\sqrt{1-k_{1,3}^2}} F(\mu, k) + \frac{\sqrt{1-k_{1,3}^2}}{k_{1,2}k_{1,3}} E(\mu, k), \tag{7}$$

where

$$\mu = \arcsin\sqrt{1-k_{1,3}^2},\quad k = \sqrt{\frac{1-k_{1,2}^2}{1-k_{1,3}^2}},$$

$F(\mu,\ k)$ and $E(\mu,\ k)$ are incomplete elliptic integrals of the first and second kind respectively.

As the auxiliary relation we may use the equations for the form factors K_1 and K_2 of an ellipsoid in regular modes of the first or the second kind:

$$\frac{1}{K_1} = \frac{2.47\,(1+0.162n_{\mathrm{E}})(n_{\mathrm{E}}+1)}{R_{\mathrm{E}}^2} = \frac{3.29}{c^2}\left(1+k_{1,2}^2+k_{1,3}^2\right), \tag{8}$$

$$\frac{1}{K_2} = \frac{(3+n_{\mathrm{E}})(n_{\mathrm{E}}+1)}{R_{\mathrm{E}}^2} = \frac{5}{c^2}\left(1+k_{1,2}^2+k_{1,3}^2\right). \tag{9}$$

Selecting, for example, Eq. (9) and solving it simultaneously with (5), we obtain

$$\frac{R_{\mathrm{E}}}{c} = \frac{4}{3} \cdot \frac{1}{(\varphi_{\mathrm{E}}-1)\,k_{1,2}k_{1,3}f(k_{1,2};\ k_{1,3})}, \tag{10}$$

$$n_{\mathrm{E}} = \frac{3-\varphi_{\mathrm{E}}}{\varphi_{\mathrm{E}}-1},\quad \varphi_{\mathrm{E}} = \frac{20}{9} \cdot \frac{1+k_{1,2}^2+k_{1,3}^2}{[k_{1,2}k_{1,3}f(k_{1,2};\ k_{1,3})]^2}. \tag{11}$$

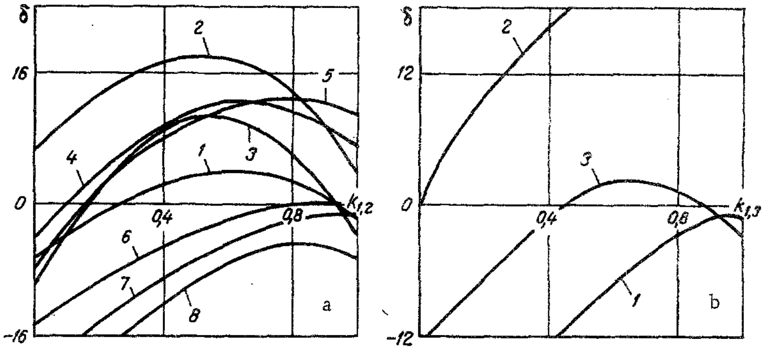

Fig. 1. Relative error in calculating the center temperature of: (a) a parallelepiped with $k_{1,3} = 1.0$ (1), 0.5 (2), 0.1 (3) on the basis of formula (12) ($\Phi(k_{1,2};\ k_{1,3}) = \Phi_0$); 1.0 (4), 0.5 (5), 0.1 (6) on the basis of an elliptical cylinder, 1.0 (7), 0.5 (8) on the basis of an ellipsoid; (b) a finite-length cylinder according to formula (12) (1), on the basis of an elliptical cylinder (2), on the basis of an ellipsoid (R = C) (3).

Solving the system of Eqs. (1)-(4) and (10), we find the sought dimension R and factor n of the real body

$$R = \sqrt{S}\,\Phi(k_{1,2};\ k_{1,3}),\quad n = \frac{S\sqrt{S}}{V}\Phi(k_{1,2};\ k_{1,3}) - 1. \tag{12}$$

Values of the function $\Phi(k_{1,2};\ k_{1,3})$

$$\Phi(k_{1,2};\ k_{1,3}) = \frac{4}{3\sqrt{2\pi}} \cdot \frac{1}{(\varphi_E - 1)\,k_{1,2}k_{1,3}f(k_{1,2};\ k_{1,3})^{3/2}}, \tag{13}$$

are listed in Table 1.

If a real body belongs to the class of elliptical cylinders, then its characteristic parameters R and n are determined by the same method, with the specifics of the basic model taken into account. Equations (1) and (2) are rewritten as

$$\frac{L_1}{c} = \frac{L_2}{b},\quad R = R_c\,(c \leqslant b,\ L_1 \leqslant L_2); \tag{14}$$

and Eq. (4) remains valid with respect to the real body and the basic model. Instead of equating the surface areas, we stipulate equal perimeters of the cross sections F and F_c normal to the longest dimension $2L_3$ of the real body and to the axis of the elliptical cylinder respectively:

$$F = F_c. \tag{15}$$

In order to find the parameters R_c and n_c of the elliptical cylinder, we modify Eqs. (5), (8), and (9) as follows:

$$\frac{R_c}{c} = \frac{\pi}{4} \cdot \frac{n_c + 1}{f(k_{1,2})},\quad k_{1,2} = \frac{c}{b} = \frac{L_1}{L_2}; \tag{16}$$

$$\frac{1}{K_1} = \frac{2.47\,(1 + 0.162 n_c)(n_c + 1)}{R_c^2} = \frac{2.89}{c^2}\left(1 + k_{1,2}^2\right); \tag{17}$$

$$\frac{1}{K_2} = \frac{(3 + n_c)(n_c + 1)}{R_c^2} = \frac{4}{c^2}\left(1 + k_{1,2}^2\right), \tag{18}$$

where $f(k_{1,2}) = E(\sqrt{1 - k_{1,2}^2})$ is a complete elliptic integral of the second kind.

From Eqs. (16) and (18) follows

$$\frac{R_c}{c} = \frac{\pi}{2} \cdot \frac{1}{(\varphi_c - 1) f(k_{1,2})},\quad n_c = \frac{3 - \varphi_c}{\varphi_c - 1}, \tag{19}$$

where

$$\varphi_c = \frac{\pi^2}{4} \cdot \frac{1 + k_{1,2}^2}{f^2(k_{1,2})}. \tag{20}$$

TABLE 3. Values of the Functions $\Phi_0(k_{1,2};\ k_{1,3})$

$k_{1,2}$	$k_{1,3}$										
	1	0,9	0,8	0,7	0,6	0,5	0,4	0,3	0,2	0,1	0
1	0,283	0,272	0,261	0,248	0,233	0,216	0,196	0,171	0,141	0,100	0
0,9	0,272	0,263	0,253	0,240	0,226	0,209	0,189	0,166	0,137	0,098	0
0,8	0,261	0,253	0,242	0,230	0,217	0,202	0,183	0,161	0,133	0,094	0
0,7	0,248	0,240	0,230	0,220	0,207	0,193	0,175	0,154	0,127	0,091	0
0,6	0,233	0,226	0,217	0,207	0,196	0,182	0,166	0,146	0,121	0,086	0
0,5	0,216	0,209	0,202	0,193	0,182	0,170	0,155	0,136	0,113	0,081	0
0,4	0,196	0,189	0,183	0,175	0,166	0,155	0,141	0,125	0,104	0,074	0
0,3	0,171	0,166	0,161	0,154	0,146	0,136	0,125	0,110	0,092	0,066	0
0,2	0,141	0,137	0,133	0,127	0,121	0,113	0 104	0,092	0,076	0,055	0
0,1	0,100	0,098	0,094	0,091	0,086	0,081	0,074	0,066	0,055	0,039	0
0	0	0	0	0	0	0	0	0	0	0	0

Parameters R and n of a real body in the class of elliptical cylinders are found by solving the system of Eqs. (14), (15), (19) and then from the following relations:

$$R = F\Phi(k_{1,2}),\ n = \frac{SF}{V}\Phi(k_{1,2}) - 1, \tag{21}$$

with V and S denoting the volume and the surface area of the real body. Values of the function $\Phi(k_{1,2})$

$$\Phi(k_{1,2}) \simeq \frac{\pi}{8}\cdot\frac{k_{1,2}}{(\varphi_c - 1)f^2(k_{1,2})}, \tag{22}$$

are listed in Table 2.

If a real body belongs to the class of plates, then its parameters are determined from the formulas

$$R = L_1,\ n = \frac{SL_1}{V} - 1, \tag{23}$$

where L_1, S, and V denote its smallest half-dimension, its surface area, and its volume respectively.

There are no precise criteria for classifying bodies into ellipsoids, elliptical cylinders, and plates. For this reason, the applicability ranges of the proposed formulas (12), (21), and (23) are defined on the basis of a comparison between exact and approximate temperature values for bodies in the shape of parallelepipeds and finite-length cylinders with sides ratios varying within wide limits. The steady-state mean-over-the-surface and center temperatures were calculated for the case with a uniform distribution of heat sources over the body volume and with the coefficient of heat transfer from the surface assumed infinite [8].

On this basis, we make the following recommendations. When the ratios (6) of orthogonal linear segments (dimensions) of a body are

$$0{,}4 \leqslant k_{1,2} \leqslant 1,\ 0.4 \leqslant k_{1,3} \leqslant 1, \tag{24}$$

then the body is to be classified as an ellipsoid with the characteristic dimension and the form factor calculated according to Eq. (12).

When the dimensions ratios are

$$0{,}2 \leqslant k_{1,2} \leqslant 1,\ 0 \leqslant k_{1,3} \leqslant 0{,}4 \tag{25}$$

then the body is to be classified as an elliptical cylinder with R and n calculated according to Eq. (21).

When

$$0 \leqslant k_{1,2} \leqslant 0{,}2,\ 0 \leqslant k_{1,3} \leqslant 0{,}2, \tag{26}$$

then the body is to be classified as a plate and formula (23) will apply. Thus, with a choice of linear segments satisfying the condition

$$2L_1 \leqslant 2L_2 \leqslant 2L_3,$$

inequalities (24)-(26) cover the entire range of possible deformations of a real body with the dimensions ratios varying within the square

$$0 \leqslant k_{1,2} \leqslant 1,\ 0 \leqslant k_{1,3} \leqslant 1.$$

An examination of the results has shown that, when conditions (24)-(26) prevail, the maximum difference between exact and approximate values of the steady-state temperature for parallelepipeds (Fig. 1a) and for finite-length cylinders (Fig. 1b) occur at the boundary between ranges (24) and (25): up to 16% for the center temperature and 10% for the mean-over-the-volume temperature, becoming much less farther away from this boundary. Within range (24) the approximate center temperature is higher while the approximate mean-over-the-volume temperature is lower than the respective exact temperature. The situation reverses within range (25).

It has been established that the calculated values change slightly, if formulas (12) and (21) are derived from relations (8) and (17) instead of (9) and (18).

It is possible to establish the approximate geometrical similarity between a real body and an ellipsoid by a simpler but less accurate method. As the characteristic dimension of the body is selected the smallest semiaxis c of a triaxial ellipsoid whose surface area is equal to the surface area of the real body. Besides condition (3), one also retains relations (1) and (4). Under assumptions made for the calculation of parameters R and n, we obtain formulas of the (12) kind with $\Phi(k_{1,2};\ k_{1,3})$ replaced by a new function (Table 3)

$$\Phi_0(k_{1,2};\ k_{1,3}) = \frac{1}{\sqrt{2\pi f(k_{1,2};\ k_{1,3})}}, \tag{27}$$

and $f(k_{1,2};\ k_{1,3})$ defined by relation (7).

Expressions (12) and (27) are valid for any dimensions ratios $0 \le k_{1,2} \le 1$ and $0 \le k_{1,3} \le 1$.

A comparison between exact and approximate values of the steady-state temperature in bodies considered here indicates that the maximum error in calculating the center and the mean-over-the-volume temperature is not greater than 25% when $k_{1,2} \approx 0.4$-0.3 and $k_{1,3} \approx 0.4$-0.3, then decreases fast at other values of these ratios (Fig. 1). It must be noted that an ellipsoid selected as one reference model does not yield precise transitions to the extreme forms of cylinder and plate. In those latter cases the error in calculating the center temperature does not exceed 7-12%.

Thus, the sequence of steps in determining the characteristic dimension and the form factor of a body reduces to: deriving relations (6) from the given surface area S and volume V of the real body with characteristic dimensions $2L_1$, $2L_2$, $2L_3$, then classifying the body into (24), (25), or (26). Depending on the class, one uses formula (12), (21), or (23) for calculations. A rougher estimate of R and n can be made according to formulas (12) and (27) at any dimensions ratios $k_{1,2}$ and $k_{1,3}$.

LITERATURE CITED

1. É. M. Gol'dberg, "Scientific reports from universities," Metallurgiya, 1, No. 3 (1958).
2. A. G. Temkin, Izv. VUZov SSSR Énergetika, 4, No. 3 (1961).
3. M. D. Mikhailov, Transient Heat and Mass Transfer in One-Dimensional Bodies [in Russian], Izd. ITMO Akad. Nauk BSSR, Minsk (1969).
4. N. A. Yaryshev, in: Heat and Mass Transfer [in Russian], Izd. Nauka i Tekh., Minsk (1968), Vol. 8.
5. N. A. Yaryshev, Inzh. Fiz. Zh., 18, No. 5 (1970).
6. A. V. Lykov, Theory of Heat Conduction [in Russian], Gostekhizdat (1952).
7. G. M. Kondrat'ev, Regular Thermal Process Mode [in Russian], Gostekhteorizdat (1954).
8. N. A. Yaryshev, Izv. VUZov SSSR Priborostroenie, 13, No. 3 (1970).